The Stretch Factor of the Delaunay Triangulation Is Less Than 1.998*

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Abstract

Let S be a finite set of points in the Euclidean plane. Let D be a Delaunay triangulation of S. The *stretch factor* (also known as *dilation* or *spanning ratio*) of D is the maximum ratio, among all points p and q in S, of the shortest path distance from p to q in D over the Euclidean distance $||pq||$. Proving a tight bound on the stretch factor of the Delaunay triangulation has been a long standing open problem in computational geometry.

In this paper we prove that the stretch factor of the Delaunay triangulation of a set of points in the plane is less than $\rho = 1.998$, improving the previous best upper bound of 2.42 by Keil and Gutwin (1989). Our bound 1.998 is better than the current upper bound of 2.33 for the special case when the point set is in convex position by Cui, Kanj and Xia (2009). This upper bound breaks the barrier 2, which is significant because previously no family of plane graphs was known to have a stretch factor guaranteed to be less than 2 on any set of points.

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1 Introduction

Let S be a finite set of points in the Euclidean plane. A *Delaunay triangulation* of S is a triangulation in which the circumscribed circle of every triangle contains no point of S in its interior. An alternative equivalent definition is: An edge xy is in the Delaunay triangulation of S if and only if there exists a circle through points x and y whose interior is devoid of points of S. A Delaunay triangulation of S is the dual graph of the Voronoi diagram of S.

Let D be a Delaunay triangulation of S. The *stretch factor* (also known as *dilation* or *spanning ratio*) of D is the maximum ratio, among all points p and q in S, of the shortest path distance from p to q in D over the Euclidean distance $||pq||$.

Proving a tight bound on the stretch factor of the Delaunay triangulation has been a long standing open problem in computational geometry. Chew [5] showed a lower bound of $\pi/2$ on the stretch factor of the Delaunay triangulation. This lower bound of $\pi/2$ was widely believed to be tight until recently (2009) when Bose et al. [2] proved that the lower bound is at least $1.5846 > \pi/2$, which was further improved to 1.5907 by Xia and Zhang [17]. Dobkin, Friedman, and Supowit [7, 8] in 1987 showed that the Delaunay triangulation has stretch factor at most $(1 + \sqrt{5})\pi/2 \approx 5.08$. This upper bound was improved by Keil and Gutwin [12, 13] in 1989 to $2\pi/(3\cos(\pi/6)) \approx 2.42$, which currently stands as the best upper bound on the stretch factor of the Delaunay triangulation. For the special case when the point set S is in convex position, Cui, Kanj and Xia [6] recently proved that the Delaunay triangulation of S has stretch factor at most 2.33.

In addition to its theoretical interest, improving the upper bound on the stretch factor of the Delaunay triangulation has a direct impact on the problem of constructing geometric *spanners* of Euclidean graphs, which has significant applications in the area of wireless computing (for more details, see [15]). Many spanner constructions in the literature rely on extracting subgraphs of the Delaunay triangulation (see for example [3, 9, 10, 14]) and their spanning ratio

is expressed as a function of the stretch factor of the Delaunay triangulation. Hence improving the upper bound on the stretch factor of the Delaunay triangulation automatically improves the upper bounds on the spanning ratio of all such spanners.

In this paper we show that the stretch factor of the Delaunay triangulation of a point set in the plane is less than $\rho = 1.998$, improving the current best upper bound of 2.42 by Keil and Gutwin [12, 13]. Our bound 1.998 is better than the current upper bound of 2.33 for the special case when the point set is in convex position [6].

The notion of stretch factor can be defined on any plane graph G as the maximum ratio, among all vertices u and v in G, of the shortest path distance between u and v in G over the Euclidean distance $||uv||$. The family of plane graphs with the best known upper bound on the stretch factor on any set of points has a stretch factor of 2 [4, 5]. Our result improves this bound to 1.998.

Our approach in proving the upper bound on the stretch factor of the Delaunay triangulation is significantly different from the previous approaches [7, 8, 12, 13, 6]. Our approach focuses on the geometry of a "chain" of circles in the plane. The "stretch factor" of a chain can be defined in analogy to that of the Delaunay triangulation. We prove an upper bound on the stretch factor of a chain and derive the same upper bound on the stretch factor of the Delaunay triangulation as a special case.

The paper is organized as follows. The necessary definitions are given in Section 2. The main theorem of the paper is given in Section 3. There we also discuss the proof strategy and provide an outline of the proof for the main theorem. Most of the technical details are captured by two lemmas, whose proofs appear in Section 4 and Section 5. The paper ends with a discussion on the possible improvements of our approach in Section 6.

2 Preliminaries

We label the points in the plane by lower case letters, such as p, q, u, v, etc. For any two points p, q in the plane, denote by pq a line in the plane passing through p and q, by $\overline{pq}$ the line segment connecting p and q, and by $\overrightarrow{pq}$ the ray from p to q. The Euclidean distance between p and q is denoted by $||pq||$. The length of a path P in the plane is denoted by $|P|$. Any angle denoted by $\angle poq$ is measured from $\overrightarrow{op}$ to $\overrightarrow{oq}$ in the *counterclockwise* direction. Unless otherwise specified, the angles are defined in the range $(-\pi, \pi]$.

Definition 1. We say that a sequence of distinct finite circles[1] $\mathcal{O} = (O_1, O_2, \ldots, O_n)$ in the plane is a *chain* if it has the following two properties. **Property (1)**: Every two consecutive circles O_i, O_{i+1} intersect, $1 \leq i \leq n-1$. Let a_i and b_i be the shared points on their boundary (in the special case where O_i, O_{i+1} are tangent, $a_i = b_i$). Without loss of generality, assume a_i's are on one side of $\mathcal{O}$ and b_i's are on the other side. Denote by $C_i^{(i+1)}$ the arc on the boundary of O_i that is in O_{i+1}, and by $C_{i+1}^{(i)}$ the arc on the boundary of O_{i+1} that is in O_i. We refer to $C_i^{(i+1)}$ and $C_{i+1}^{(i)}$ as "connecting arcs". **Property (2)**: The connecting arcs $C_i^{(i-1)}$ and $C_i^{(i+1)}$ on the same circle O_i do not overlap, for $2 \leq i \leq n-1$; i.e., $C_i^{(i-1)}$ and $C_i^{(i+1)}$ do not share any point other than a boundary point. See Figure 1 for an illustration. Denote by $\mathcal{O}_{i,j}$ a sub-chain of $\mathcal{O}$: $\mathcal{O}_{i,j} = (O_i, \ldots, O_j)$.

Given a chain $\mathcal{O} = (O_1, O_2, \ldots, O_n)$. Let u be a point on the boundary of O_1 that is not in the interior of O_2. Let v be a point on the boundary of O_n that is not in the interior of O_{n-1}. We call u, v *a pair of terminal points* (or simply *terminals*) of the chain $\mathcal{O}$. For notational convenience, define $a_0 = b_0 = u$ and $a_n = b_n = v$. Every circle O_i has two arcs on its boundary between the line segments $\overline{a_{i-1}b_{i-1}}$ and $\overline{a_i b_i}$, denoted by A_i and B_i. Without loss of generality, assume that a_{i-1}, a_i are the ends of A_i and b_{i-1}, b_i are the ends of B_i, for $1 \leq i \leq n$. This means that $A_1 \ldots A_n$ is a path from u to v on one side of the

[1] In this paper, a circle is considered to be a closed disk in the plane

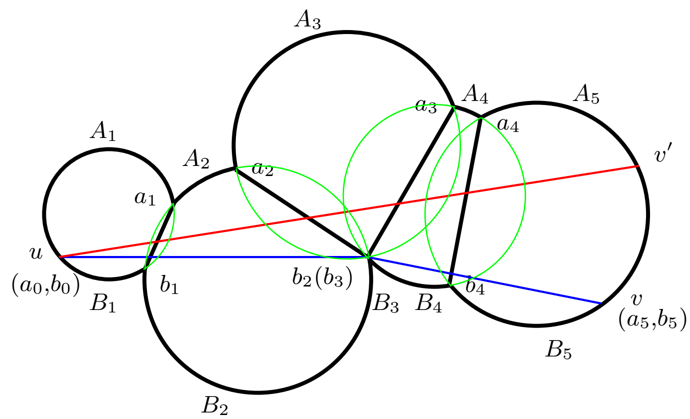

Figure 1: A chain $\mathcal{O}$. The connecting arcs are green (gray in black and white printing). The connecting arcs on the boundary of the same circle are disjoint. In this example, the terminals u, v are obstructed while the terminals u, v' are unobstructed.

chain and $B_1 \dots B_n$ is a path from u to v on the other side of the chain. An arc A_i or B_i may degenerate to a point, in which case $a_{i-1} = a_i$ or $b_{i-1} = b_i$, respectively.

Let $D_{\mathcal{O}}(u, v) = up_1 \dots p_{n-1}v$ be the shortest polyline from u to v that consists of line segments $\overline{up_1}, \overline{p_1p_2}, \dots, \overline{p_{n-1}v}$, where $p_i \in \overline{a_ib_i}$ for $1 \le i \le n-1$. The length of $D_{\mathcal{O}}(u, v)$, denoted by $|D_{\mathcal{O}}(u, v)|$, is the sum of the lengths of the line segments in $D_{\mathcal{O}}(u, v)$. See Figure 1. If the polyline $D_{\mathcal{O}}(u, v)$ contains a point p_j which is a_j or b_j, for some $1 \le j \le n-1$, we say that u, v are *obstructed*. If u, v are unobstructed, then $D_{\mathcal{O}}(u, v)$ is the line segment $\overline{uv}$. We have the following fact.

Fact 1. *If $D_{\mathcal{O}}(u, v)$ is the line segment $\overline{uv}$, then $\overrightarrow{uv}$ stabs $O_1, \dots, O_n$ in order. That is, for any $1 \le i < j \le n$, $\overrightarrow{uv}$ enters O_i no later than entering O_j and exits O_i no later than exiting O_j.*

Proof. For $2 \le i \le n-1$, the two connecting arcs on O_i do not overlap. Hence $\overline{a_{i-1}b_{i-1}}$ and $\overline{a_{i+1}b_{i+1}}$ must appear on the different sides of the line a_ib_i. Without loss of generality, assume that a_ib_i is a vertical line which divides the

plane into two half-planes, $\overline{a_{i-1}b_{i-1}}$ appears in the left half-plane and $\overline{a_{i+1}b_{i+1}}$ appears in the right half-plane. If $D_{\mathcal{O}}(u,v)$ is the line segment $\overline{uv}$, then $\overrightarrow{uv}$ intersects $\overline{a_ib_i}$ from left to right. Thus $\overrightarrow{uv}$ enters O_{i-1} and O_i in the left half-plane. Since O_i is contained in O_{i-1} in the left half-plane, $\overrightarrow{uv}$ must enter O_{i-1} no later than entering O_i. Likewise, $\overrightarrow{uv}$ exits O_{i-1} and O_i in the right half-plane. Since O_{i-1} is contained in O_i in the right half-plane, $\overrightarrow{uv}$ must exit O_{i-1} no later than exiting O_i. This completes the proof. $\square$

The polyline $D_{\mathcal{O}}(u,v)$ can be visualized as a "rubber band" connecting u and v "inside" the circles in $\mathcal{O}$. The polyline $D_{\mathcal{O}}(u,v)$ can be regarded as a correspondence of the straight line between two point u,v in the Delaunay triangulation. Note that for the purpose of bounding the stretch factor of the Delaunay triangulation, only the case where u,v are unobstructed is relevant. However, for our proof to work, it is necessary to consider the case when u,v are obstructed (see Section 4).

We define the shortest path between u and v in $\mathcal{O}$, denoted by $P_{\mathcal{O}}(u,v)$, to be the shortest path from u to v while traveling along arcs in $\{A_1,\ldots,A_n\} \cup \{B_1,\ldots,B_n\}$ and line segments in $\{\overline{a_1b_1},\ldots,\overline{a_{n-1}b_{n-1}}\}$. Note that the vertex set of $P_{\mathcal{O}}(u,v)$ is a subset of $\{a_0,\ldots,a_n\} \cup \{b_0,\ldots,b_n\}$ and the edge set of $P_{\mathcal{O}}(u,v)$ is a subset of $\{A_1,\ldots,A_n\} \cup \{B_1,\ldots,B_n\} \cup \{\overline{a_1b_1},\ldots,\overline{a_{n-1}b_{n-1}}\}$. Its length, $|P_{\mathcal{O}}(u,v)|$, is the total length of the edges in $P_{\mathcal{O}}(u,v)$. For example, in Figure 1, $P_{\mathcal{O}}(u,v)$ is the shortest path from u to v while traveling along the thick arcs and lines. $P_{\mathcal{O}}(u,v)$ corresponds to the shortest path between two point u,v in the Delaunay triangulation.

Now we can define the *stretch factor* of a chain $\mathcal{O}$, denoted by $C_{\mathcal{O}}$, as the maximum value of

$$|P_{\mathcal{O}}(u,v)|/|D_{\mathcal{O}}(u,v)|, \tag{1}$$

over all terminals u,v of $\mathcal{O}$. The stretch factor of a chain is analogous to that of a Delaunay triangulation.

In this paper, we will focus our attention on the stretch factor of a chain. Compared to the Delaunay triangulation, it is much easier to manipulate a

chain. For example, it is easy to transform a single circle in a chain without affecting other circles in the chain. In contrast, changing a triangle in a Delaunay triangulation always affects other triangles.

3 The Main Theorem

The following is the main theorem of this paper, which gives an upper bound on the stretch factor of a chain.

Theorem 1. *For all $\mathcal{O}$, the stretch factor $C_{\mathcal{O}}$ is less than ρ, where $\rho = 1.998$.*

Assuming Theorem 1 is true, we can derive an improved upper bound on the stretch factor of the Delaunay triangulation as a special case.

Corollary 1. *The stretch factor of a Delaunay triangulation D of a set of points S in the plane is less than $\rho = 1.998$.*

Proof. For any two points $x, y \in S$. Let $\mathcal{T}$ be the sequence of triangles in D crossed by $\overrightarrow{xy}$. Let $\mathcal{O}$ be the corresponding sequence of circumscribed circles of the triangles in $\mathcal{T}$. It is clear that $\mathcal{O}$ is a chain and x, y are terminals of $\mathcal{O}$. The shortest path distance from x to y is at most $|P_{\mathcal{O}}(x, y)|$ and $||xy|| = |D_{\mathcal{O}}(x, y)|$. By Theorem 1, the stretch factor of a Delaunay triangulation is at most $|P_{\mathcal{O}}(x, y)|/||xy|| = |P_{\mathcal{O}}(x, y)|/|D_{\mathcal{O}}(x, y)| < \rho$. $\square$

Due to the complex nature of the proof for Theorem 1, in this section we discuss the proof strategy and present a outline of the proof. The main technical details of the proof are captured in two lemmas, whose proofs are given in the subsequent sections.

When $\mathcal{O}$ has only one circle, it is easy to see that for all $\mathcal{O}, u$, and v, $|P_{\mathcal{O}}(u, v)|/|D_{\mathcal{O}}(u, v)| \leq \pi/2 < \rho$. So it is natural to attempt an inductive proof based on the number of circles in $\mathcal{O}$. A simple induction would require us to show that adding a circle to a chain will not increase the stretch factor. However, this does not work because one can always increase the stretch factor of a chain by adding a circle to it, albeit by a very small amount [17]. We

tackle this problem by amortized analysis. Specifically, we introduce a potential function $\Phi_{\mathcal{O}}$ and define a target function $\Upsilon_{\mathcal{O}}(u, v)$:

$$\Upsilon_{\mathcal{O}}(u, v) = |P_{\mathcal{O}}(u, v)| - \lambda |D_{\mathcal{O}}(u, v)| + \Phi_{\mathcal{O}}, \tag{2}$$

where $\lambda = 1.8$ is a parameter whose value is determined by the potential function. The goal is to prove Theorem 1 by showing $\Upsilon_{\mathcal{O}}(u, v) < 0$ for all $\mathcal{O}, u$, and v.

The key component of our strategy is the selection of an appropriate potential function $\Phi_{\mathcal{O}}$, which is described in the following.

Definition 2. Let O_{i-1} and O_i be two consecutive circles in a chain $\mathcal{O}$. Without loss of generality, assume that their centers o_{i-1}, o_i lie on a horizontal line and that a_{i-1} is on or above the line $o_{i-1}o_i$. Refer to Figure 2. Let $q_{i-1}^{\rightarrow}$ be the "peak" of O_{i-1} with regard to $o_{i-1}o_i$, i.e., the point on the upper boundary of O_{i-1} that is farthest from the line $o_{i-1}o_i$. Likewise, let $q_i^{\leftarrow}$ be the "peak" of O_i with regard to $o_{i-1}o_i$ (the sign $^{\rightarrow}$ or $^{\leftarrow}$ indicates whether the peak is defined with the previous circle or the next circle in $\mathcal{O}$). Let $Q_{i-1}^{\rightarrow}$ be the upper arc between $q_{i-1}^{\rightarrow}$ and a_{i-1} on the boundary of O_{i-1} and let $Q_i^{\leftarrow}$ be the upper arc between $q_i^{\leftarrow}$ and a_{i-1} on the boundary of O_i. If $Q_{i-1}^{\rightarrow}$ is inside O_i, we color it *red*; otherwise we color $Q_{i-1}^{\rightarrow}$ *green*. Likewise, we color $Q_i^{\leftarrow}$ red or green depending on whether it is inside O_{i-1}. Let $\mathcal{P}_i$ be a path from $q_{i-1}^{\rightarrow}$ to $q_i^{\leftarrow}$ consisting of $Q_{i-1}^{\rightarrow}$ and $Q_i^{\leftarrow}$. Let H_i be the horizontal distance traveled along the path $\mathcal{P}_i$ with green arcs contributing positively to H_i and red arcs contributing negatively to H_i. Similarly, let V_i be the vertical distance traveled along the path $\mathcal{P}_i$ with green arcs contributing positively to V_i and red arcs contributing negatively to V_i. The potential function is defined as follows:

$$\Phi_{\mathcal{O}} = \varphi(r_n - r_1) - \frac{\varphi}{3} \sum_{i=2}^{n} (2H_i + V_i), \tag{3}$$

where r_i is the radius of O_i and $\varphi = \frac{3}{\sqrt{5}}(1 - \lambda/\rho)$ is a parameter that determines the "weight" of the potential function.

We have the following fact.

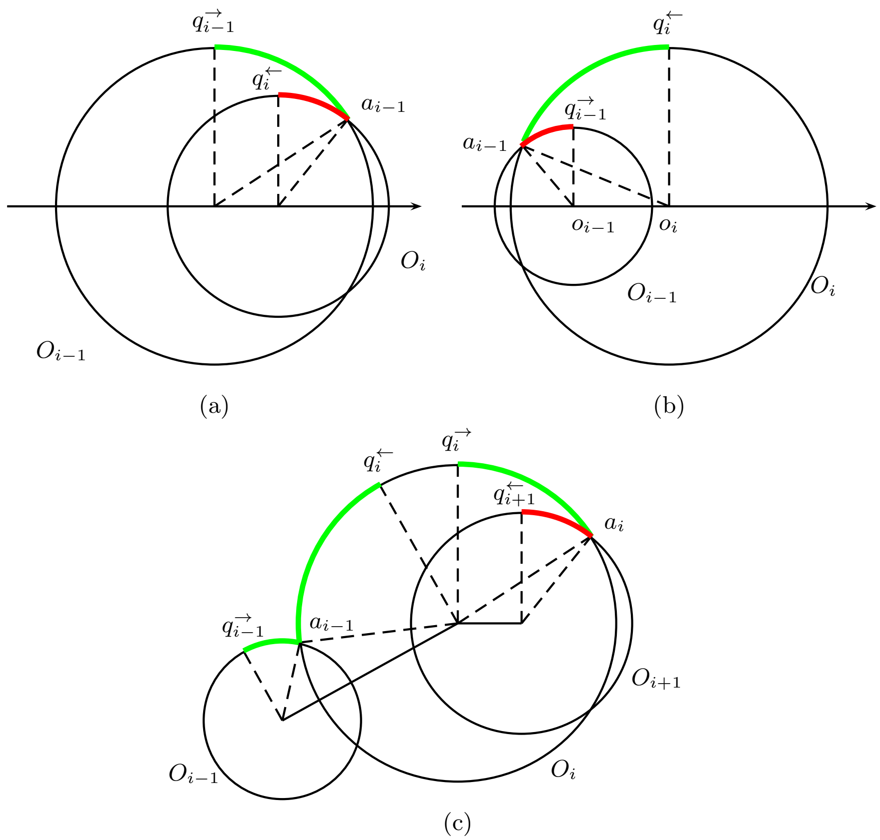

Figure 2: The definition of $\mathcal{P}_i$—the thick colored arcs (in black and white printing, red is the darker color and green is the lighter color). In (a) and (b) $\mathcal{P}_i$ has one green arc and one red arc. In (c) $\mathcal{P}_i$ has two green arcs. (c) shows an example of $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ defined on three consecutive circles O_{i-1}, O_i and O_{i+1}. Note that O_i has two different peaks. The peak q_i^- is defined with the previous circle O_{i-1}. The peak q_i^+ is defined with the next circle O_{i+1}.

Fact 2. *Let r_i and r_{i-1} be the radii of O_i and O_{i-1}, respectively. Then $H_i = ||o_i o_{i-1}||$ and $V_i \geq |r_i - r_{i-1}|$.*

Proof. For this proof, fix a coordinate system where the origin is o_i, x-axis is $\overrightarrow{o_{i-1}o_i}$, and a_{i-1} is above the x-axis. Let $X_{q^{\rightarrow}_{i-1}}$ and $Y_{q^{\rightarrow}_{i-1}}$ be the x- and y-coordinates of $q^{\rightarrow}_{i-1}$. Let $X_{q^{\leftarrow}_i}$ and $Y_{q^{\leftarrow}_i}$ be the x- and y-coordinates of $q^{\leftarrow}_i$. Let $X_{a_{i-1}}$ and $Y_{a_{i-1}}$ be the x- and y-coordinates of a_{i-1}. We distinguish three cases.

- Case 1. $Q^{\rightarrow}_{i-1}$ is green and $Q^{\leftarrow}_i$ is red. See Figure 2(a). Then

  $$H_i = (X_{a_{i-1}} - X_{q^{\rightarrow}_{i-1}}) - (X_{a_{i-1}} - X_{q^{\leftarrow}_i}) = X_{q^{\leftarrow}_i} - X_{q^{\rightarrow}_{i-1}} = ||o_i o_{i-1}||.$$

  In this case, $r_i < r_{i-1}$. We have

  $$V_i = (Y_{q^{\rightarrow}_{i-1}} - Y_{a_{i-1}}) - (Y_{q^{\leftarrow}_i} - Y_{a_{i-1}}) \geq Y_{q^{\rightarrow}_{i-1}} - Y_{q^{\leftarrow}_i} = r_{i-1} - r_i = |r_i - r_{i-1}|.$$

- Case 2. $Q^{\rightarrow}_{i-1}$ is red and $Q^{\leftarrow}_i$ is green. See Figure 2(b). Then

  $$H_i = (X_{q^{\leftarrow}_i} - X_{a_{i-1}}) - (X_{q^{\rightarrow}_{i-1}} - X_{a_{i-1}}) = X_{q^{\leftarrow}_i} - X_{q^{\rightarrow}_{i-1}} = ||o_i o_{i-1}||.$$

  In this case, $r_i > r_{i-1}$. We have

  $$V_i = (Y_{q^{\leftarrow}_i} - Y_{a_{i-1}}) - (Y_{q^{\rightarrow}_{i-1}} - Y_{a_{i-1}}) \geq Y_{q^{\leftarrow}_i} - Y_{q^{\rightarrow}_{i-1}} = r_i - r_{i-1} = |r_i - r_{i-1}|.$$

- Case 3. Both $Q^{\rightarrow}_{i-1}$ and $Q^{\leftarrow}_i$ are green. See Figure 2(c). Then

  $$H_i = (X_{q^{\leftarrow}_i} - X_{a_{i-1}}) + (X_{a_{i-1}} - X_{q^{\rightarrow}_{i-1}}) = X_{q^{\leftarrow}_i} - X_{q^{\rightarrow}_{i-1}} = ||o_i o_{i-1}||, \text{ and}$$
  $$V_i = (Y_{q^{\leftarrow}_i} - Y_{a_{i-1}}) + (Y_{q^{\rightarrow}_{i-1}} - Y_{a_{i-1}}) \geq |Y_{q^{\leftarrow}_i} - Y_{q^{\rightarrow}_{i-1}}| = |r_i - r_{i-1}|.$$

So in all cases, the statement is true. This completes the proof. □

The potential function is designed with three goals in mind.

First, the potential function $\Phi_{\mathcal{O}}$ is designed such that adding a circle to $\mathcal{O}$ does not increase $\Phi_{\mathcal{O}}$, as shown below.

Proposition 1. $\Phi_{\mathcal{O}} \leq \Phi_{\mathcal{O}_{1,n-1}}$.

Proof. By the triangle inequality, $||o_n o_{n-1}|| \geq ||o_n a_{n-1}|| - ||o_{n-1} a_{n-1}|| = r_n - r_{n-1}$. Combining this with Fact 2, we have

$$\begin{aligned} 2H_n + V_n &\geq 2||o_n o_{n-1}|| + |r_n - r_{n-1}| \\ &\geq 2(r_n - r_{n-1}) + (r_n - r_{n-1}) \\ &= 3(r_n - r_{n-1}). \end{aligned} \tag{4}$$

So $\Phi_{\mathcal{O}} - \Phi_{\mathcal{O}_{1,n-1}} = \varphi(r_n - r_{n-1}) - \frac{\varphi}{3}(2H_n + V_n) \leq 0$. $\square$

Secondly, the potential function $\Phi_{\mathcal{O}}$ is designed such that $\Upsilon_{\mathcal{O}}(u, v) < 0$ for all $\mathcal{O}, u$, and v. This is Lemma 1, the main technical lemma of the paper, whose proof is given in Section 4.

Lemma 1. *For all $\mathcal{O}, u$, and v, $\Upsilon_{\mathcal{O}}(u, v) < 0$.*

Thirdly, the potential function $\Phi_{\mathcal{O}}$ is designed such that its value can be bounded from below as a function of $|P_{\mathcal{O}}(u, v)|$ for some chain $\mathcal{O}$ with certain extremal properties. This is Lemma 2, whose proof appears in Section 5.

Lemma 2. *Let $\mathbb{O}$ be a set of chains whose stretch factor is greater than or equal to a threshold τ. If $\mathbb{O}$ is not empty, then there exists a chain $\mathcal{O}^* \in \mathbb{O}$ with terminals u, v such that $|P_{\mathcal{O}^*}(u, v)|/|D_{\mathcal{O}^*}(u, v)| \geq \tau$ and $\Phi_{\mathcal{O}^*} \geq -\frac{\sqrt{5}\varphi}{3}|P_{\mathcal{O}^*}(u, v)|$.*

With Lemma 1 and Lemma 2, we are ready to give the proof for the main theorem.

(proof of Theorem 1). For sake of contradiction, suppose that there is a non-empty set $\mathbb{O}$ of chains $\mathcal{O}$ with stretch factor $C_{\mathcal{O}} \geq \rho$. By Lemma 2, there exists a chain $\mathcal{O}^* \in \mathbb{O}$ with terminals u and v such that

$$|P_{\mathcal{O}^*}(u, v)|/|D_{\mathcal{O}^*}(u, v)| \geq \rho, \tag{5}$$

and

$$\Phi_{\mathcal{O}^*} \geq -\frac{\sqrt{5}\varphi}{3}|P_{\mathcal{O}^*}(u, v)|. \tag{6}$$

By Lemma 1,

$$\Upsilon_{\mathcal{O}^*}(u,v) = |P_{\mathcal{O}^*}(u,v)| - \lambda|D_{\mathcal{O}^*}(u,v)| + \Phi_{\mathcal{O}^*} < 0. \tag{7}$$

Combining (6) and (7), we have

$$\begin{aligned}
&|P_{\mathcal{O}^*}(u,v)| - \lambda|D_{\mathcal{O}^*}(u,v)| - \frac{\sqrt{5}\varphi}{3}|P_{\mathcal{O}^*}(u,v)| \\
&\le |P_{\mathcal{O}^*}(u,v)| - \lambda|D_{\mathcal{O}^*}(u,v)| + \Phi_{\mathcal{O}^*} \\
&= \Upsilon_{\mathcal{O}^*}(u,v) < 0.
\end{aligned} \tag{8}$$

Recall that $\varphi = \frac{3}{\sqrt{5}}(1-\lambda/\rho)$. Rearranging (8), we have

$$\begin{aligned}
|P_{\mathcal{O}^*}(u,v)| &< \frac{\lambda}{1-\frac{\sqrt{5}\varphi}{3}}|D_{\mathcal{O}^*}(u,v)| \\
&= \frac{\lambda}{1-(1-\lambda/\rho)}|D_{\mathcal{O}^*}(u,v)| \\
&= \rho|D_{\mathcal{O}^*}(u,v)|.
\end{aligned} \tag{9}$$

This is a contradiction to (5). Therefore $C_{\mathcal{O}} < \rho$ for all $\mathcal{O}$. □

The rest of the paper contains the proofs for Lemma 1 and Lemma 2.

4 Proof of Lemma 1

This section is devoted to the proof of Lemma 1.

(proof of Lemma 1). Proceed by induction on n, the number of circles in $\mathcal{O}$. If $n = 1$, then $\Phi_{\mathcal{O}} = 0$, $P_{\mathcal{O}}(u,v)$ is the shorter arc between u and v on the boundary of O_1, and $D_{\mathcal{O}}(u,v)$ is the chord between u and v inside O_1. So $|P_{\mathcal{O}}(u,v)| \le \pi/2|D_{\mathcal{O}}(u,v)|$ and hence $\Upsilon_{\mathcal{O}}(u,v) = |P_{\mathcal{O}}(u,v)| - \lambda|D_{\mathcal{O}}(u,v)| + \Phi_{\mathcal{O}} = |P_{\mathcal{O}}(u,v)| - \lambda|D_{\mathcal{O}}(u,v)| < 0$.

Now assuming that the statement is true when there are less than n circles in $\mathcal{O}$, we will prove that it is true when there are n circles in $\mathcal{O}$.

For the rest of this section, fix an arbitrary chain $\mathcal{O} = (O_1, O_2, \ldots, O_n)$ and an arbitrary terminal u on the boundary of O_1. We start by considering some special cases.

First consider the case when v is either a_{n-1} or b_{n-1}.

Proposition 2. *$\Upsilon_{\mathcal{O}}(u, a_{n-1}) < 0$ and $\Upsilon_{\mathcal{O}}(u, b_{n-1}) < 0$.*

Proof. Consider the sub-chain $\mathcal{O}_{1,n-1} = (O_1, \ldots, O_{n-1})$. Since u, a_{n-1} are terminals of $\mathcal{O}_{1,n-1}$, by the inductive hypothesis,

$$\begin{aligned}
&\Upsilon_{\mathcal{O}_{1,n-1}}(u, a_{n-1}) \\
&= |P_{\mathcal{O}_{1,n-1}}(u, a_{n-1})| - \lambda |D_{\mathcal{O}_{1,n-1}}(u, a_{n-1})| + \Phi_{\mathcal{O}_{1,n-1}} \\
&< 0. \qquad (10)
\end{aligned}$$

It is easy to verify that $|P_{\mathcal{O}}(u, a_{n-1})| \leq |P_{\mathcal{O}_{1,n-1}}(u, a_{n-1})|$ and $|D_{\mathcal{O}}(u, a_{n-1})| = |D_{\mathcal{O}_{1,n-1}}(u, a_{n-1})|$. Combining these with (10), we have

$$\begin{aligned}
\Upsilon_{\mathcal{O}}(u, a_{n-1}) &= |P_{\mathcal{O}}(u, a_{n-1})| - \lambda |D_{\mathcal{O}}(u, a_{n-1})| + \Phi_{\mathcal{O}} \\
&\leq |P_{\mathcal{O}_{1,n-1}}(u, a_{n-1})| - \lambda |D_{\mathcal{O}_{1,n-1}}(u, a_{n-1})| + \Phi_{\mathcal{O}} \\
&\leq \Upsilon_{\mathcal{O}_{1,n-1}}(u, a_{n-1}) + \Phi_{\mathcal{O}} - \Phi_{\mathcal{O}_{1,n-1}} \\
&< \Phi_{\mathcal{O}} - \Phi_{\mathcal{O}_{1,n-1}} \\
&\leq 0. \qquad (11)
\end{aligned}$$

The last inequality is from Proposition 1. Similarly, we have $\Upsilon_{\mathcal{O}}(u, b_{n-1}) < 0$. ◻

Next consider the case when u and v are obstructed.

Proposition 3. *If u and v are obstructed, then $\Upsilon_{\mathcal{O}}(u, v) < 0$.*

Proof. If u and v are obstructed, then $D_{\mathcal{O}}(u, v)$ contains a point p_j that is either a_j or b_j, for some $1 \leq j \leq n-1$. Without loss of generality, assume $p_j = a_j$. Consider two sub-chains of $\mathcal{O}$: $\mathcal{O}_{1,j+1} = (O_1, \ldots, O_{j+1})$ and $\mathcal{O}_{j+1,n} = (O_{j+1}, \ldots, O_n)$. Points u, a_j are terminals of $\mathcal{O}_{1,j+1}$ and points a_j, v are terminals of $\mathcal{O}_{j+1,n}$. If $j = n-1$, by Proposition 2,

$$\Upsilon_{\mathcal{O}_{1,j+1}}(u, a_j) = \Upsilon_{\mathcal{O}}(u, a_{n-1}) < 0. \qquad (12)$$

If $j < n-1$, $\mathcal{O}_{1,j+1}$ has less than n circles, and by the inductive hypothesis

$$\Upsilon_{\mathcal{O}_{1,j+1}}(u, a_j) < 0. \tag{13}$$

On the other hand, $\mathcal{O}_{j+1,n}$ has less than n circles, and by the inductive hypothesis,

$$\Upsilon_{\mathcal{O}_{j+1,n}}(a_j, v) < 0. \tag{14}$$

Note that

$$|P_{\mathcal{O}}(u,v)| \leq |P_{\mathcal{O}_{1,j+1}}(u, a_j)| + |P_{\mathcal{O}_{j+1,n}}(a_j, v)|, \tag{15}$$

$$|D_{\mathcal{O}}(u,v)| = |D_{\mathcal{O}_{1,j+1}}(u, a_j)| + |D_{\mathcal{O}_{j+1,n}}(a_j, v)|, \tag{16}$$

and

$$\Phi_{\mathcal{O}} = \Phi_{\mathcal{O}_{1,j+1}} + \Phi_{\mathcal{O}_{j+1,n}}. \tag{17}$$

Combining (12)—(17), we have

$$\begin{aligned}
\Upsilon_{\mathcal{O}}(u,v) &= |P_{\mathcal{O}}(u,v)| - \lambda |D_{\mathcal{O}}(u,v)| + \Phi_{\mathcal{O}} \\
&\leq |P_{\mathcal{O}_{1,j+1}}(u, a_j)| - \lambda |D_{\mathcal{O}_{1,j+1}}(u, a_j)| + \Phi_{\mathcal{O}_{1,j+1}} \\
&\quad + |P_{\mathcal{O}_{j+1,n}}(a_j, v)| - \lambda |D_{\mathcal{O}_{j+1,n}}(a_j, v)| + \Phi_{\mathcal{O}_{j+1,n}} \\
&= \Upsilon_{\mathcal{O}_{1,j+1}}(u, a_j) + \Upsilon_{\mathcal{O}_{j+1,n}}(a_j, v) \\
&< 0,
\end{aligned} \tag{18}$$

as desired. □

Now we only need to consider the set of unobstructed terminal points v on O_n, which form an arc, denoted by $\widehat{A}$ and referred to as the *unobstructed arc*[2]. Denote by $P_{\mathcal{O}}^{A_n}(u,v)$ and $P_{\mathcal{O}}^{B_n}(u,v)$ the shortest paths from u to v that includes A_n and B_n (on O_n), respectively. We call v *pivotal* if $|P_{\mathcal{O}}^{A_n}(u,v)| = |P_{\mathcal{O}}^{B_n}(u,v)|$.

[2]For notational convenience, assume that $\widehat{A}$ includes its boundary points. This makes $\widehat{A}$ a closed set.

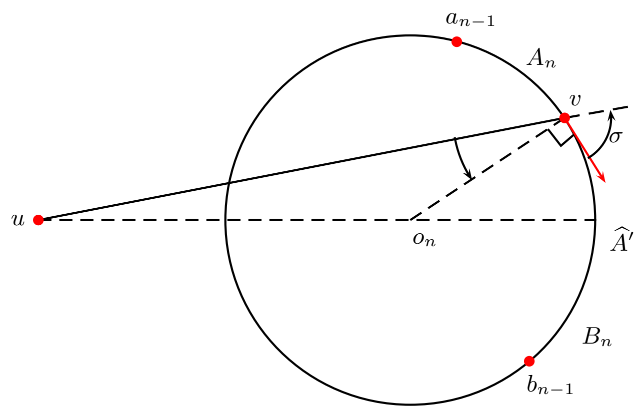

Figure 3: Illustration for Proposition 4. The angle $\angle uvo_n$ is currently in range $(0, \pi/2)$. As v moves away from a_{n-1} along $\widehat{A}'$, $\angle uvo_n$ decreases. When v moves below uo_n, $\angle uvo_n$ becomes negative.

Proposition 4. *The maximum of $\Upsilon_{\mathcal{O}}(u,v)$ occurs when v is on the boundary of $\widehat{A}$ or when v is pivotal in $\widehat{A}$.*

Proof. Let $\widehat{A}'$ be an arbitrary sub-arc of $\widehat{A}$ that does not contain a pivotal point in its interior. In order to prove the proposition, it suffices to show that the maximum of $\Upsilon_{\mathcal{O}}(u,v)$ for all $v \in \widehat{A}'$ occurs when v is on the boundary of $\widehat{A}'$.

Since $\widehat{A}'$ does not contain a pivotal point in its interior, either $|P_{\mathcal{O}}^{A_n}(u,v)| \leq |P_{\mathcal{O}}^{B_n}(u,v)|$ for every point $v \in \widehat{A}'$ or $|P_{\mathcal{O}}^{B_n}(u,v)| \leq |P_{\mathcal{O}}^{A_n}(u,v)|$ for every point $v \in \widehat{A}'$. Without loss of generality assume that $|P_{\mathcal{O}}^{A_n}(u,v)| \leq |P_{\mathcal{O}}^{B_n}(u,v)|$ for every point v in $\widehat{A}'$. Fixing other parameters, $\Upsilon_{\mathcal{O}}(u,v)$ is a function of $|A_n|$ as v moves along $\widehat{A}'$. One observes the following. First, $\Phi_{\mathcal{O}}$ remains constant when v moves along $\widehat{A}'$. Secondly, $|P_{\mathcal{O}}(u,v)|$ is a linear function of $|A_n|$ because $|P_{\mathcal{O}}(u,v)| = |P_{\mathcal{O}}^{A_n}(u,v)| = |P_{\mathcal{O}}(u,a_{n-1})| + |A_n|$, where $|P_{\mathcal{O}}(u,a_{n-1})|$ remains constant when v moves along $\widehat{A}'$. Thirdly, $-\lambda|D_{\mathcal{O}}(u,v)|$ is a convex function of $|A_n|$. To see why, refer to Figure 3. One observes that

$$\frac{d|D_{\mathcal{O}}(u,v)|}{d|A_n|} = \cos\sigma = \sin(\angle uvo_n). \tag{19}$$

Also observe that $\angle uvo_n$ decreases as v moves away from a_{n-1} along $\widehat{A}'$ and hence $\frac{d\angle uvo_n}{d|A_n|} \leq 0$. Since v is the exit point of ray $\overrightarrow{uv}$ on O_n, $\angle uvo_n \in (-\pi/2, \pi/2)$ and hence $\frac{d\sin(\angle uvo_n)}{d\angle uvo_n} \geq 0$. So $\frac{d\sin(\angle uvo_n)}{d|A_n|} = \frac{d\sin(\angle uvo_n)}{d\angle uvo_n} \cdot \frac{d\angle uvo_n}{d|A_n|} \leq 0$. Combining this with (19), we have

$$\frac{d^2(-\lambda|D_{\mathcal{O}}(u,v)|)}{d|A_n|^2} = \frac{d(-\lambda\sin(\angle uvo_n))}{d|A_n|} \geq 0,$$

which implies that $-\lambda|D_{\mathcal{O}}(u,v)|$ is a convex function of $|A_n|$.

Being a sum of convex functions, $\Upsilon_{\mathcal{O}}(u,v) = |P_{\mathcal{O}}(u,v)| - \lambda|D_{\mathcal{O}}(u,v)| + \Phi_{\mathcal{O}}$ is also a convex function of $|A_n|$. This proves that the maximum of $\Upsilon_{\mathcal{O}}(u,v)$ appears on the boundary of $\widehat{A}'$, as desired. $\square$

Therefore, if we can show that $\Upsilon_{\mathcal{O}}(u,v) < 0$ when v is on the boundary of the $\widehat{A}$ and when v is pivotal, then $\Upsilon_{\mathcal{O}}(u,v) < 0$ for any point v in $\widehat{A}$ and we are done.

Proposition 5. $\Upsilon_{\mathcal{O}}(u,v) < 0$ *if v is on the boundary of $\widehat{A}$.*

Proof. If $v = a_{n-1}$ or $v = b_{n-1}$, by Proposition 2, $\Upsilon_{\mathcal{O}}(u,v) < 0$. If v is neither a_{n-1} nor b_{n-1}, then u and v must be obstructed and by Proposition 3, $\Upsilon_{\mathcal{O}}(u,v) < 0$. $\square$

What remains to be shown is that $\Upsilon_{\mathcal{O}}(u,v) < 0$ when v is pivotal. This requires a careful analysis of the geometry. To start, we fix a coordinate system where the origin is the center point of $\overline{a_{n-1}b_{n-1}}$, the x-axis is $o_{n-1}o_n$ where o_{n-1} is to the left of o_n, and the y-axis is $a_{n-1}b_{n-1}$ where a_{n-1} is above b_{n-1}. See Figure 4 for an illustration. Since $\overrightarrow{uv}$ crosses $\overline{a_{n-1}b_{n-1}}$ from left to right, u is to the left of the y-axis and v is to the right of the y-axis. By flipping the geometry along the x-axis, if necessary, we can assume $Y_v \leq Y_u$, where Y_v and Y_u are the y-coordinates of v and u, respectively. Let q be the exit point of the x-axis on O_n. Let v' be the entry-point of $\overrightarrow{uv}$ on O_n.

We define the following parameters. Let $\alpha = \angle qo_na_{n-1}$ and $\beta = \angle vo_nq$. Let γ be the angle from $\overrightarrow{uv}$ to the x-axis in the *counterclockwise* direction. Their ranges are given as follows.

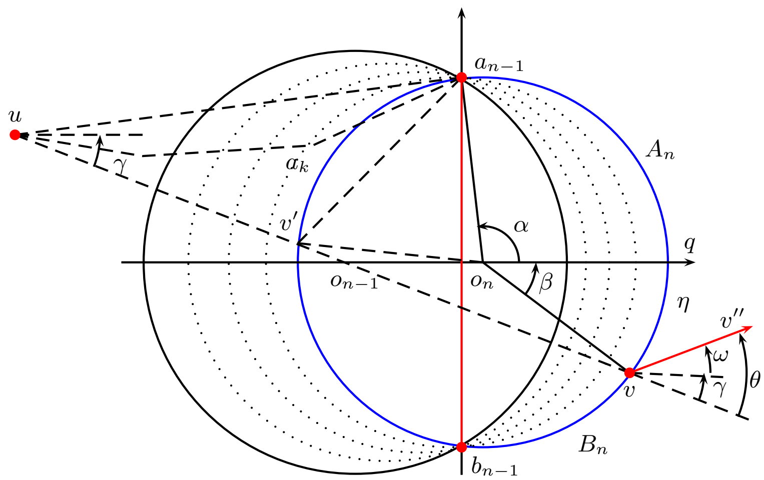

Figure 4: Illustration of the coordinate system, the definition of the parameters α, β, γ, and the movement of o_n toward o_{n-1}.

Since a_{n-1} is on or above the x-axis, $0 \le \alpha \le \pi$. If $\alpha = 0$ or $\alpha = \pi$, then $a_{n-1} = b_{n-1}$, which means that $D_{\mathcal{O}}(u, v)$ contains a_{n-1} and $\Upsilon_{\mathcal{O}}(u, v) < 0$ by Proposition 3. So we can assume

$$0 < \alpha < \pi. \tag{20}$$

Since v is a pivotal point, $|P_{\mathcal{O}}^{A_n}(u, v)| = |P_{\mathcal{O}}^{B_n}(u, v)|$, where $|P_{\mathcal{O}}^{A_n}(u, v)| = |P_{\mathcal{O}}(u, a_{n-1})| + |A_n|$ and $|P_{\mathcal{O}}^{B_n}(u, v)| = |P_{\mathcal{O}}(u, b_{n-1})| + |B_n|$. Since a_{n-1} and b_{n-1} are connected by a line segment $\overline{a_{n-1}b_{n-1}}$, we have $-||a_{n-1}b_{n-1}|| \le |P_{\mathcal{O}}(u, a_{n-1})| - |P_{\mathcal{O}}(u, b_{n-1})| \le ||a_{n-1}b_{n-1}||$. So $-||a_{n-1}b_{n-1}|| \le |A_n| - |B_n| \le ||a_{n-1}b_{n-1}||$, where $||a_{n-1}b_{n-1}|| = 2r_n \sin\alpha$ and $|A_n| - |B_n| = 2r_n\beta$. Thus the range of β is

$$-\sin\alpha \le \beta \le \sin\alpha. \tag{21}$$

We have $0 \le \gamma$ because $Y_v \le Y_u$. Since uv crosses $\overline{a_{n-1}b_{n-1}}$, the largest value of γ occurs when $\overline{uv}$ passes through a_{n-1}. This means $\gamma \le \pi/2 - \angle b_{n-1}a_{n-1}v =$

$\pi/2 - (\alpha - \beta)/2$, since $\angle b_{n-1}a_{n-1}v = (\alpha - \beta)/2$. So the range of γ is

$$0 \le \gamma \le \pi/2 - (\alpha - \beta)/2 < \pi/2. \tag{22}$$

The last inequality is true because $\alpha - \beta \ge \alpha - \sin\alpha > 0$.

We proceed by distinguishing two cases depending on the value of γ in comparison to a threshold value γ^+ defined as

$$\gamma^+ = \frac{3\sin\alpha - \alpha}{4} + \arcsin\left(\frac{\alpha + \sin\alpha}{4\lambda\sin(\frac{\alpha+\sin\alpha}{4})}\right). \tag{23}$$

Proposition 6. *If v is a pivotal point and $\gamma \ge \gamma^+$, then $\Upsilon_{\mathcal{O}}(u, v) < 0$.*

Proof. Since u, v are unobstructed, $\overrightarrow{uv}$ crosses all line segments $\overline{a_i b_i}$, $1 \le i \le n-1$. The points $\{a_1, \ldots, a_{n-1}\}$ are all above the line uv. Recall that $D_{\mathcal{O}}(u, v)$ is a "rubber band" between u and v. When one end of the "rubber band" moves from v to a_{n-1} along the boundary of O_n, the result is polyline $D_{\mathcal{O}}(u, a_{n-1})$, which is a path from u to a_{n-1} that is convex-away from ua_{n-1} and is between $\overrightarrow{ua_{n-1}}$ and $\overrightarrow{uv}$.

See Figure 4. Let a_k be the last turning point in the polyline $D_{\mathcal{O}}(u, a_{n-1})$. Then the part of $D_{\mathcal{O}}(u, a_{n-1})$ between a_k and a_{n-1} is a line segment, and a_k, a_{n-1} are terminals of the sub-chain $\mathcal{O}_{k+1,n}$. By Fact 1, $\overrightarrow{a_k a_{n-1}}$ stabs $O_{k+1}, \ldots, O_n$ in order. Since $a_k \ne a_{k+1}$ (otherwise a_{k+1} would be the last turning point), a_k is an entry point on O_{k+1}. This means that a_k appears no later than the entry point of $D_{\mathcal{O}}(u, a_{n-1})$ on O_n. Combining this with the fact that $D_{\mathcal{O}}(u, a_{n-1})$ is between $\overrightarrow{ua_{n-1}}$ and $\overrightarrow{uv}$, we know that a_k is in the triangle $\triangle uv'a_{n-1}$. Now $\overline{ua_{n-1}} \cup D_{\mathcal{O}}(u, a_{n-1})$ is a convex polygon which contains $\overline{a_k a_{n-1}}$ as an edge. So the whole path $D_{\mathcal{O}}(u, a_{n-1})$ is above the line $a_k a_{n-1}$. This implies that $D_{\mathcal{O}}(u, a_{n-1})$ lies in the triangle $\triangle uv'a_{n-1}$. See Figure 4 for an illustration.

Since $D_{\mathcal{O}}(u, a_{n-1})$ is a path from u to a_{n-1} that is convex-away from ua_{n-1} in the triangle $\triangle uv'a_{n-1}$. By convexity, we have $|D_{\mathcal{O}}(u, a_{n-1})| \le ||uv'|| + ||v'a_{n-1}||$ (see [1, p. 42]). Recall that $|D_{\mathcal{O}}(u, v)| = ||uv||$. So

$$\begin{aligned} |D_{\mathcal{O}}(u, v)| - |D_{\mathcal{O}}(u, a_{n-1})| &\ge ||uv|| - (||uv'|| + ||v'a_{n-1}||) \\ &= ||v'v|| - ||v'a_{n-1}||. \end{aligned} \tag{24}$$

Refer to Figure 4, $||v'v|| = 2r_n \cos(\angle vv'o_n)$ and $||v'a_{n-1}|| = 2r_n \cos(\angle o_n v' a_{n-1})$, where $\angle vv'o_n = \angle o_n vv' = \beta - \gamma$ and $\angle o_n v' a_{n-1} = \angle vv'a_{n-1} - \angle vv'o_n = (\alpha+\beta)/2 - (\beta - \gamma) = \alpha/2 - \beta/2 + \gamma$. From (24) and by the trigonometric identity, we have

$$\begin{aligned}
&|D_{\mathcal{O}}(u,v)| - |D_{\mathcal{O}}(u,a_{n-1})| \\
&\geq ||v'v|| - ||v'a_{n-1}|| \\
&= 2r_n \cos(\beta-\gamma) - 2r_n \cos(\alpha/2 - \beta/2 + \gamma) \\
&= -4r_n \sin(\alpha/4+\beta/4) \sin(3\beta/4 - \alpha/4 - \gamma). \qquad (25)
\end{aligned}$$

Since v is pivotal, $|P_{\mathcal{O}}(u,v)| = |P_{\mathcal{O}}(u,a_{n-1})| + |A_n|$, where $|A_n| = r_n(\alpha+\beta)$. By Proposition 2, $\Upsilon_{\mathcal{O}}(u,a_{n-1}) < 0$. Combining these with (25), we have

$$\begin{aligned}
\Upsilon_{\mathcal{O}}(u,v) &= |P_{\mathcal{O}}(u,v)| - \lambda|D_{\mathcal{O}}(u,v)| + \Phi_{\mathcal{O}} \\
&= |P_{\mathcal{O}}(u,a_{n-1})| + |A_n| - \lambda|D_{\mathcal{O}}(u,v)| + \Phi_{\mathcal{O}} \\
&= \Upsilon_{\mathcal{O}}(u,a_{n-1}) + |A_n| - \lambda(|D_{\mathcal{O}}(u,v)| - |D_{\mathcal{O}}(u,a_{n-1})|) \\
&< |A_n| - \lambda(|D_{\mathcal{O}}(u,v)| - |D_{\mathcal{O}}(u,a_{n-1})|) \\
&\leq r_n(\alpha+\beta) + 4r_n\lambda \sin(\alpha/4+\beta/4)\sin(3\beta/4 - \alpha/4 - \gamma). \qquad (26)
\end{aligned}$$

Define a function

$$h(\alpha,\beta,\gamma) = r_n(\alpha+\beta) + 4r_n\lambda \sin(\alpha/4+\beta/4)\sin(3\beta/4-\alpha/4-\gamma).$$

Then

$$\frac{\partial h}{\partial \gamma} = -4r_n\lambda \sin(\alpha/4+\beta/4)\cos(3\beta/4-\alpha/4-\gamma).$$

By (21), we have $\alpha/4+\beta/4 \geq \alpha/4 - \sin\alpha/4 > 0$, and $\alpha/4+\beta/4 \leq \alpha/4 + \sin\alpha/4 < \pi/4$. So

$$0 < \alpha/4+\beta/4 < \pi/4. \qquad (27)$$

By (22), we have $3\beta/4 - \alpha/4 - \gamma \geq 3\beta/4 - \alpha/4 - \pi/2 + (\alpha-\beta)/2 = \alpha/4+\beta/4 - \pi/2 > -\pi/2$. Also $3\beta/4 - \alpha/4 - \gamma < 3\beta/4 \leq 3/4$. So

$$-\pi/2 < 3\beta/4 - \alpha/4 - \gamma < 3/4. \qquad (28)$$

From (27) and (28), $\sin(\alpha/4+\beta/4) > 0$ and $\cos(3\beta/4-\alpha/4-\gamma) > 0$, and hence

$$\frac{\partial h}{\partial \gamma} < 0. \tag{29}$$

Let

$$\gamma^*(\alpha,\beta) = 3\beta/4 - \alpha/4 + \arcsin\left(\frac{\alpha+\beta}{4\lambda\sin(\alpha/4+\beta/4)}\right).$$

Then $h(\alpha,\beta,\gamma^*) = 0$. Let $\mu = \alpha/4+\beta/4$ and $\nu = \frac{\mu}{\lambda\sin\mu}$. By (27), $0 < \mu < \pi/4$. We have

$$\frac{\partial \mu}{\partial \beta} = \frac{\partial(\alpha/4+\beta/4)}{\partial \beta} > 0. \tag{30}$$

Since $\mu < \tan\mu$ and $\sin\mu > 0$ for $0 < \mu < \pi/4$, we have

$$\frac{\partial \nu}{\partial \mu} = \frac{\partial \frac{\mu}{\lambda\sin\mu}}{\partial \mu} = (1-\mu/\tan\mu)/(\lambda\sin\mu) > 0. \tag{31}$$

From (31), $\nu = \frac{\mu}{\lambda\sin\mu} \le \frac{\pi/4}{\lambda\sin(\pi/4)} < 0.618$. Also $\nu > 0$. So

$$\frac{\partial \arcsin\nu}{\partial \nu} = 1/\sqrt{1-\nu^2} > 0, \tag{32}$$

By (30), (31), and (32),

$$\frac{\partial \gamma^*}{\partial \beta} = 3/4 + \frac{\partial \arcsin\nu}{\partial \nu}\cdot\frac{\partial \nu}{\partial \mu}\cdot\frac{\partial \mu}{\partial \beta} > 0.$$

This means γ^* is increasing in β, and hence

$$\gamma^*(\alpha,\beta) \le \gamma^*(\alpha,\sin\alpha) = \gamma^+.$$

By (29), h is decreasing in γ. So for any $\gamma \ge \gamma^+ \ge \gamma^*$, we have

$$h(\alpha,\beta,\gamma) \le h(\alpha,\beta,\gamma^+) \le h(\alpha,\beta,\gamma^*) = 0.$$

Combining this with (26), we have $\Upsilon_{\mathcal{O}}(u,v) < h(\alpha,\beta,\gamma) \le 0$ as desired. □

Proposition 7. *If v is a pivotal point and $\gamma < \gamma^+$, then $\Upsilon_{\mathcal{O}}(u,v) < 0$.*

Proof. We perform the following transformation. Move o_n toward o_{n-1} (the center of O_{n-1}), along the x-axis while keeping a_{n-1} and b_{n-1} on the boundary of O_n until either $\gamma \ge \gamma^+$ or o_n reaches o_{n-1}. In the transformation, v stays

pivotal and the radius of O_n changes. See Figure 4. The dotted circles show the process of the transformation. At the end of the transformation, if $\gamma \geq \gamma^+$, then $\Upsilon_{\mathcal{O}}(u, v) < 0$ by Proposition 6, and if o_n reaches o_{n-1}, then $\Upsilon_{\mathcal{O}}(u, v) = \Upsilon_{\mathcal{O}_{1,n-1}}(u, v) < 0$ by the inductive hypothesis. Therefore, in order to prove that $\Upsilon_{\mathcal{O}}(u, v) < 0$ before the transformation, it suffices to show that $\Upsilon_{\mathcal{O}}(u, v)$ does not decrease during the transformation when X_{o_n}, the x-coordinate of o_n, decreases (i.e., we only need to show that $\frac{\partial \Upsilon_{\mathcal{O}}(u,v)}{\partial X_{o_n}} \leq 0$).

Refer to Figure 4. Let X_u, Y_u and X_v, Y_v be the x- and y-coordinates of u and v, respectively. Let $Y_{a_{n-1}}$ be the y-coordinate of a_{n-1}. Note that X_u, Y_u, and $Y_{a_{n-1}}$ remain constant during the transformation. Let $\eta = \beta r_n$. Since v stays pivotal during the transformation, $\eta = (A_n - B_n)/2 = (|P_{\mathcal{O}}(u, b_{n-1})| - |P_{\mathcal{O}}(u, a_{n-1})|)/2$ remains constant during the transformation.

In the following, we express all other parameters as functions of (X_{o_n}, η, X_u, Y_u, $Y_{a_{n-1}}$) and then calculate their partial derivatives with respect to X_{o_n}. This is possible because as mentioned above, η, X_u, Y_u, and $Y_{a_{n-1}}$ are independent of X_{o_n}. Note that $\sin\alpha = \frac{Y_{a_{n-1}}}{r_n}$ and $\cos\alpha = -\frac{X_{o_n}}{r_n}$.

$$\frac{\partial r_n}{\partial X_{o_n}} = \frac{\partial \sqrt{X_{o_n}^2 + Y_{a_{n-1}}^2}}{\partial X_{o_n}} = \frac{X_{o_n}}{\sqrt{X_{o_n}^2 + Y_{a_{n-1}}^2}} = \frac{X_{o_n}}{r_n} = -\cos\alpha. \tag{33}$$

$$\frac{\partial \alpha}{\partial X_{o_n}} = \frac{\partial(\pi/2 + \arctan(\frac{X_{o_n}}{Y_{a_{n-1}}}))}{\partial X_{o_n}} = \frac{Y_{a_{n-1}}}{X_{o_n}^2 + Y_{a_{n-1}}^2} = \frac{Y_{a_{n-1}}}{r_n^2} = \frac{\sin\alpha}{r_n}. \tag{34}$$

$$\frac{\partial(\alpha r_n)}{\partial X_{o_n}} = \alpha\frac{\partial r_n}{\partial X_{o_n}} + r_n\frac{\partial \alpha}{\partial X_{o_n}} = \sin\alpha - \alpha\cos\alpha. \tag{35}$$

$$\frac{\partial \beta}{\partial X_{o_n}} = \frac{\partial(\eta/r_n)}{\partial X_{o_n}} = -\frac{\eta}{r_n^2}\frac{\partial r_n}{\partial X_{o_n}} = \frac{\beta\cos\alpha}{r_n}. \tag{36}$$

$$\begin{aligned}\frac{\partial X_v}{\partial X_{o_n}} &= \frac{\partial(X_{o_n} + r_n\cos\beta)}{\partial X_{o_n}} = 1 - r_n\sin\beta\frac{\partial \beta}{\partial X_{o_n}} + \cos\beta\frac{\partial r_n}{\partial X_{o_n}} \\ &= 1 - \beta\cos\alpha\sin\beta - \cos\alpha\cos\beta.\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{\partial Y_v}{\partial X_{o_n}} &= \frac{\partial(-r_n\sin\beta)}{\partial X_{o_n}} = -r_n\cos\beta\frac{\partial \beta}{\partial X_{o_n}} - \sin\beta\frac{\partial r_n}{\partial X_{o_n}} \\ &= -\beta\cos\alpha\cos\beta + \cos\alpha\sin\beta.\end{aligned} \tag{38}$$

Because $P_{\mathcal{O}}(u, a_{n-1})$ and η remain constant during the transformation, from

(35) we have

$$\begin{aligned}
\frac{\partial|P_{\mathcal{O}}(u,v)|}{\partial X_{o_n}} &= \frac{\partial(P_{\mathcal{O}}(u,a_{n-1})+|A_n|)}{\partial X_{o_n}} \\
&= \frac{\partial|A_n|}{\partial X_{o_n}} \\
&= \frac{\partial(\alpha r_n+\eta)}{\partial X_{o_n}} \\
&= \frac{\partial(\alpha r_n)}{\partial X_{o_n}} \\
&= \sin\alpha-\alpha\cos\alpha. \qquad (39)
\end{aligned}$$

From Fact 2, $H_n = ||o_n o_{n-1}|| = X_{o_n} - X_{o_{n-1}}$. Since $X_{o_{n-1}}$ is constant during the transformation,

$$\frac{\partial H_n}{\partial X_{o_n}} = \frac{\partial X_{o_n}}{\partial X_{o_n}} = 1.$$

The vertical distance traveled by $Q_n^{\leftarrow}$ is $r_n - Y_{a_{n-1}}$ and the vertical distance traveled by $Q_{n-1}^{\rightarrow}$ is $r_{n-1} - Y_{a_{n-1}}$. When $0 < \alpha < \pi/2$, $Q_n^{\leftarrow}$ is inside O_{n-1} and hence is red. So $V_n = (r_{n-1} - Y_{a_{n-1}}) - (r_n - Y_{a_{n-1}}) = r_{n-1} - r_n$. Since r_{n-1} is constant during the transformation,

$$\frac{\partial V_n}{\partial X_{o_n}} = -\frac{\partial r_n}{\partial X_{o_n}} = \cos\alpha = |\cos\alpha|.$$

When $\pi/2 \le \alpha < \pi$, both $Q_n^{\leftarrow}$ and $Q_{n-1}^{\rightarrow}$ are green. So $V_n = (r_{n-1} - Y_{a_{n-1}}) + (r_n - Y_{a_{n-1}})$. Since r_{n-1} and $Y_{a_{n-1}}$ are constant during the transformation,

$$\frac{\partial V_n}{\partial X_{o_n}} = \frac{\partial r_n}{\partial X_{o_n}} = -\cos\alpha = |\cos\alpha|.$$

Therefore, we have

$$\begin{aligned}
\frac{\partial \Phi_{\mathcal{O}}}{\partial X_{o_n}} &= \frac{\partial(\varphi(r_n-r_1)-\frac{\varphi}{3}\sum_{i=2}^{n-1}(2H_i+V_i))}{\partial X_{o_n}} \\
&= \varphi\frac{\partial r_n}{\partial X_{o_n}} - \frac{\varphi}{3}(2\frac{\partial H_n}{\partial X_{o_n}} + \frac{\partial V_n}{\partial X_{o_n}}) \\
&= -\varphi\cos\alpha - \frac{2\varphi}{3} - \frac{\varphi}{3}|\cos\alpha|. \qquad (40)
\end{aligned}$$

Note that $\sin\gamma = \frac{Y_u - Y_v}{||uv||}$ and $\cos\gamma = \frac{X_v - X_u}{||uv||}$. We have

$$\begin{aligned}
\frac{\partial |D_{\mathcal{O}}(u,v)|}{\partial X_{o_n}} = \frac{\partial ||uv||}{\partial X_{o_n}} &= \frac{\partial \sqrt{(X_v - X_u)^2 + (Y_v - Y_u)^2}}{\partial X_{o_n}} \\
&= \frac{X_v - X_u}{||uv||} \cdot \frac{\partial (X_v - X_u)}{\partial X_{o_n}} + \frac{Y_v - Y_u}{||uv||} \cdot \frac{\partial (Y_v - Y_u)}{\partial X_{o_n}} \\
&= \cos\gamma \frac{\partial X_v}{\partial X_{o_n}} - \sin\gamma \frac{\partial Y_v}{\partial X_{o_n}} \qquad (41) \\
&= \cos\gamma (1 - \beta \cos\alpha \sin\beta - \cos\alpha \cos\beta) \\
&\quad - \sin\gamma(-\beta\cos\alpha\cos\beta + \cos\alpha\sin\beta) \\
&= \cos\gamma - \cos\alpha(\beta\sin\beta\cos\gamma - \beta\cos\beta\sin\gamma \\
&\quad + \cos\beta\cos\gamma + \sin\beta\sin\gamma) \\
&= \cos\gamma - \cos\alpha(\cos(\beta-\gamma) + \beta\sin(\beta-\gamma)). \qquad (42)
\end{aligned}$$

Define a function

$$\begin{aligned}
f(\alpha,\beta,\gamma) &= -\lambda \frac{\partial |D_{\mathcal{O}}(u,v)|}{\partial X_{o_n}} \\
&= -\lambda(\cos\gamma - \cos\alpha(\cos(\beta-\gamma) + \beta\sin(\beta-\gamma))). \qquad (43)
\end{aligned}$$

From (39)—(43), we have

$$\begin{aligned}
\frac{\partial \Upsilon_{\mathcal{O}}(u,v)}{\partial X_{o_n}} &= \frac{\partial |P_{\mathcal{O}}(u,v)|}{\partial X_{o_n}} - \lambda \frac{\partial |D_{\mathcal{O}}(u,v)|}{\partial X_{o_n}} + \frac{\partial |\Phi_{\mathcal{O}}|}{\partial X_{o_n}} \\
&= \sin\alpha - \alpha\cos\alpha - \frac{2\varphi}{3} - \varphi\cos\alpha - \frac{\varphi}{3}|\cos\alpha| + f(\alpha,\beta,\gamma) \qquad (44)
\end{aligned}$$

In the following, we will bound $\frac{\partial \Upsilon_{\mathcal{O}}(u,v)}{\partial X_{o_n}}$ by single-variant functions in α.

Refer to Figure 4. Let v'' be the location of v when X_{o_n} increases by ∂X_{o_n} (i.e., o_n moves to the right by ∂X_{o_n}). Let $\partial\ell$ be the distance between v and v''. Let ω be the angle from x-axis to $\overrightarrow{vv''}$. So $\cos\omega = \frac{\partial X_v}{\partial \ell}$ and $\sin\omega = \frac{\partial Y_v}{\partial \ell}$, where X_v and Y_v are the x- and y-coordinates of v. Recall that γ is the angle from $\overrightarrow{uv}$ to the x-axis. Let $\theta = \omega + \gamma$.

From (41), we have

$$
\begin{aligned}
\frac{\partial |D_{\mathcal{O}}(u,v)|}{\partial X_{o_n}} &= \cos\gamma \frac{\partial X_v}{\partial X_{o_n}} - \sin\gamma \frac{\partial Y_v}{\partial X_{o_n}} \\
&= (\cos\gamma \frac{\partial X_v}{\partial \ell} - \sin\gamma \frac{\partial Y_v}{\partial \ell}) \frac{\partial \ell}{\partial X_{o_n}} \\
&= (\cos\gamma\cos\omega - \sin\gamma\sin\omega) \frac{\partial \ell}{\partial X_{o_n}} \\
&= \cos(\omega+\gamma) \frac{\partial \ell}{\partial X_{o_n}} \\
&= \cos\theta \frac{\partial \ell}{\partial X_{o_n}}.
\end{aligned} \tag{45}
$$

Therefore, $f = -\lambda \frac{\partial |D_{\mathcal{O}}(u,v)|}{\partial X_{o_n}} = -\lambda \cos\theta \frac{\partial \ell}{\partial X_{o_n}}$. Because $\frac{\partial \ell}{\partial X_{o_n}} > 0$ is independent of γ, when α and β are fixed the maximum of f occurs when $\cos\theta$ is minimized, i.e., when θ is minimized or maximized. Since $\theta = \omega + \gamma$ where ω is independent of γ, θ is minimized when $\gamma = 0$ and is maximized when $\gamma = \gamma^+$. Therefore

$$
f(\alpha,\beta,\gamma) \leq \max\{f(\alpha,\beta,0), f(\alpha,\beta,\gamma^+)\}, \tag{46}
$$

where $\gamma^+ = \frac{3\sin\alpha - \alpha}{4} + \arcsin\left(\frac{\alpha + \sin\alpha}{4\lambda \sin(\frac{\alpha+\sin\alpha}{4})}\right)$.

On the other hand, $\frac{\partial f}{\partial \beta} = \lambda\beta\cos\alpha\cos(\beta-\gamma)$. Since v is an exit point of $\overrightarrow{uv}$ on the boundary of O_n, $-\pi/2 \leq \beta - \gamma \leq \pi/2$ and hence $\cos(\beta-\gamma) \geq 0$. We distinguish two cases:

1. $\pi/2 \leq \alpha < \pi$. In this case, $\cos\alpha \leq 0$ and as mentioned above $\cos(\beta-\gamma) \geq 0$. So $\frac{\partial f}{\partial \beta} \geq 0$ when $\beta \leq 0$ and $\frac{\partial f}{\partial \beta} \leq 0$ when $\beta \geq 0$. This means that

$$
f(\alpha,\beta,\gamma) \leq f(\alpha,0,\gamma). \tag{47}
$$

   When $\pi/2 \leq \alpha < \pi$, we have $|\cos\alpha| = -\cos\alpha$, which, combined with (44), (46), and (47), implies that

$$
\begin{aligned}
\frac{\partial \Upsilon_{\mathcal{O}}(u,v)}{\partial X_{o_n}} \leq \sin\alpha - \alpha\cos\alpha - \frac{2\varphi}{3} - \frac{2\varphi}{3}\cos\alpha \\
+ \max\{f(\alpha,0,0), f(\alpha,0,\gamma^+)\}.
\end{aligned} \tag{48}
$$

2. $0 < \alpha < \pi/2$. In this case, $\cos\alpha \geq 0$ and as mentioned above $\cos(\beta - \gamma) \geq 0$. So $\frac{\partial f}{\partial \beta} \leq 0$ when $\beta \leq 0$, and $\frac{\partial f}{\partial \beta} \geq 0$ when $\beta \geq 0$. Observe

that $f(\alpha,\beta,\gamma) - f(\alpha,-\beta,\gamma) = 2\lambda\cos\alpha\sin\gamma(\sin\beta - \beta\cos\beta)$. By (22), $0 \le \gamma < \pi/2$ and hence $\sin\gamma \ge 0$. Also $\sin\beta - \beta\cos\beta \ge 0$ for $0 \le \beta \le 1$. Thus $f(\alpha,\beta,\gamma) \ge f(\alpha,-\beta,\gamma)$ for $0 \le \beta \le 1$. This means that

$$f(\alpha,\beta,\gamma) \le f(\alpha,|\beta|,\gamma). \tag{49}$$

Since $\frac{\partial f}{\partial \beta} \ge 0$ in the range $[|\beta|, \sin\alpha]$ and from (21), $|\beta| \le \sin\alpha$, we have

$$f(\alpha,|\beta|,\gamma) \le f(\alpha,\sin\alpha,\gamma). \tag{50}$$

From (49) and (50),

$$f(\alpha,\beta,\gamma) \le f(\alpha,\sin\alpha,\gamma). \tag{51}$$

When $0 < \alpha < \pi/2$, we have $|\cos\alpha| = \cos\alpha$, which, combined with (44), (46), and 51, implies that

$$\begin{aligned}\frac{\partial \Upsilon_{\mathcal{O}}(u,v)}{\partial X_{o_n}} \le &\sin\alpha - \alpha\cos\alpha - \frac{2\varphi}{3} - \frac{4\varphi}{3}\cos\alpha \\ &+ \max\{f(\alpha,\sin\alpha,0), f(\alpha,\sin\alpha,\gamma^{+})\}. \end{aligned} \tag{52}$$

By (48) and (52), we only need to verify the following four inequalities.

$$\begin{aligned} g_1(\alpha) &= \sin\alpha - \alpha\cos\alpha - \frac{2\varphi}{3} - \frac{2\varphi}{3}\cos\alpha + f(\alpha,0,0) \\ &< 0, \quad \text{when } \pi/2 \le \alpha < \pi. \end{aligned} \tag{53}$$

$$\begin{aligned} g_2(\alpha) &= \sin\alpha - \alpha\cos\alpha - \frac{2\varphi}{3} - \frac{2\varphi}{3}\cos\alpha + f(\alpha,0,\gamma^{+}) < \\ &< 0, \quad \text{when } \pi/2 \le \alpha < \pi. \end{aligned} \tag{54}$$

$$\begin{aligned} g_3(\alpha) &= \sin\alpha - \alpha\cos\alpha - \frac{2\varphi}{3} - \frac{4\varphi}{3}\cos\alpha + f(\alpha,\sin\alpha,0) \\ &< 0, \quad \text{when } 0 < \alpha < \pi/2. \end{aligned} \tag{55}$$

$$\begin{aligned} g_4(\alpha) &= \sin\alpha - \alpha\cos\alpha - \frac{2\varphi}{3} - \frac{4\varphi}{3}\cos\alpha + f(\alpha,\sin\alpha,\gamma^{+}) \\ &< 0, \quad \text{when } 0 < \alpha < \pi/2. \end{aligned} \tag{56}$$

Since g_1, g_2, g_3 and g_4 are smooth functions on small intervals of α, one can easily verify the above inequalities using a numerical computing software, such

as Mathematica. We give a more formal approach in the appendix. We show that g_1, g_2, g_3 and g_4 have small Lipschitz constants, and then use a program that implements a simplified Piyavskii's algorithm [16] to verify that their upper bounds are less than 0. $\square$

This completes the proof of Lemma 1. $\square$

5 Proof of Lemma 2

This section contains the proof of Lemma 2.

(proof of Lemma 2). Let $\mathbb{O}$ be a set of chains whose stretch factor is greater than or equal to a threshold τ. Suppose $\mathbb{O}$ is not empty. Let $\mathbb{E}$ be the subset of $\mathbb{O}$ consisting of chains with minimum number of circles; let it be n. For any chain $\mathcal{O} \in \mathbb{E}$, associate with it a pair of terminals u, v with the largest stretch factor among all pairs of terminals of $\mathcal{O}$. Fix a coordinate system. Without loss of generality, assume that all chains in $\mathbb{E}$ are normalized such that u is at the origin $(0,0)$ and v is at $(1,0)$. Each $\mathcal{O} \in \mathbb{E}$ is represented by a vector $\mathbf{x} \in \mathcal{R}^{3n}$ that specifies, for each of the n circles in $\mathcal{O}$, its radius and the x- and y-coordinates of its center. Since the conditions of the chain (Definition 1) and the conditions of $\mathbb{E}$ all include the boundary cases, $\mathbb{E}$ is represented by a non-empty *closed* set in $\mathcal{R}^{3n}$. Define a function $H(\mathbf{x}) = \sum_{i=1}^{n} r_i$. Since $H(\mathbf{x})$ is continuous and has limit ∞ for $||\mathbf{x}|| \to \infty$, H has a global minimum (see [11, p.60]). Let $\mathcal{O}^*$ be the chain in $\mathbb{E}$ that achieves the minimum of H. Then $\mathcal{O}^*$ satisfies three conditions: (1) $\mathcal{O}^*$ has stretch factor $\geq \tau$, (2) among all chains in $\mathbb{O}$, the number of circles in $\mathcal{O}^*$ is minimized, and (3) among all chains in $\mathbb{E}$, the sum of the radii $\sum_{O_i \in \mathcal{O}^*} r_i$ is minimized.

Let u, v be the terminals associate with $\mathcal{O}^*$ that yield the worst stretch factor. The chain $\mathcal{O}^*$ has the following properties.

Proposition 8. *u and v are unobstructed in $\mathcal{O}^*$.*

Proof. Suppose that $D_{\mathcal{O}^*}(u,v)$ contains a point p_j which is either a_j or b_j, $1 \leq j \leq n-1$. Consider two sub-chains of $\mathcal{O}^*$: $\mathcal{O}^*_{1,j} = (O_1, \ldots, O_j)$ and $\mathcal{O}^*_{j+1,n} = (O_{j+1}, \ldots, O_n)$. So u, p_j are terminals of $\mathcal{O}^*_{1,j}$ and p_j, v are terminals of $\mathcal{O}^*_{j+1,n}$. Note that $|P_{\mathcal{O}^*}(u,v)| \leq |P_{\mathcal{O}^*_{1,j}}(u,p_j)| + |P_{\mathcal{O}^*_{j+1,n}}(p_j,v)|$ and $|D_{\mathcal{O}^*}(u,v)| = |D_{\mathcal{O}^*_{1,j}}(u,p_j)| + |D_{\mathcal{O}^*_{j+1,n}}(p_j,v)|$. So we have either $|P_{\mathcal{O}^*_{1,j}}(u,p_j)| \geq \rho |D_{\mathcal{O}^*_{1,j}}(u,p_j)|$ or $|P_{\mathcal{O}^*_{j+1,n}}(p_j,v)| \geq \rho |D_{\mathcal{O}^*_{j+1,n}}(p_j,v)|$. This means that either $\mathcal{O}^*_{1,j}$ or $\mathcal{O}^*_{j+1,n}$ has stretch factor greater than or equal to that of $\mathcal{O}^*$ and has less number of circles than $\mathcal{O}^*$—a contradiction to condition (2) of $\mathcal{O}^*$. So u, v must be unobstructed in $\mathcal{O}^*$. $\square$

Proposition 9. *$B_1 B_2 \ldots B_n$ is a shortest path between u and v in $\mathcal{O}^*$ and so is $A_1 A_2 \ldots A_n$.*

Proof. Suppose that $B_1 B_2 \ldots B_n$ is not a shortest path between u and v. Let B_j be the first arc in the path $B_1 B_2 \ldots B_n$ that is not in a shortest path between u and v. This means that every shortest path between u and v contains A_j. This also means that B_{j-1} and $\overline{a_{j-1}b_{j-1}}$ belong to a shortest path between u and v. We can assume that B_j is not degenerated because otherwise $\overline{a_j b_j}$ will be a shortcut between b_{j-1} and a_j in the shortest path containing B_{j-1}, $\overline{a_{j-1}b_{j-1}}$ and A_j.

We perform a transformation that shrinks O_j as follows: keep a_{j-1} and a_j on the boundary of O_j and reduce the radius r_j by a small amount; in case where A_j is degenerated, keep a_j (a_{j-1}) on the boundary of O_j and move the center o_j by a small amount along the line segment $\overline{o_j a_j}$ toward a_j. See Figure 5 for an illustration. Such transformation can be performed while maintaining the following:

1. $\mathcal{O}^*$ remains a chain. Since a_{j-1} and a_j stay on the boundary of O_j, O_j remains intersected with O_{j-1} and O_{j+1} during the transformation. So property (1) of chain (in Definition 1) is satisfied. Shrinking O_j will shrink the connecting arcs $C^{(j)}_{j-1}$ and $C^{(j)}_{j+1}$, but does not affect $C^{(j-2)}_{j-1}$ and $C^{(j+2)}_{j+1}$. So $C^{(j-2)}_{j-1}$ and $C^{(j)}_{j-1}$ do not overlap after shrinking O_j, and neither do $C^{(j)}_{j+1}$

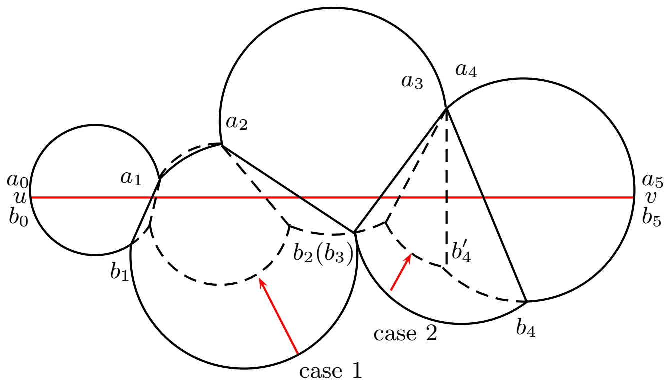

Figure 5: Illustrations of Propositions 9: shrinking a circle. Two cases are shown. A_j is not degenerated in case 1 and A_j is degenerated in case 2.

and $C_{j+1}^{(j+2)}$. The only other possible way that the property (2) of chain can be violated is for $C_j^{(j-1)}$ and $C_j^{(j+1)}$ to overlap; but we know that B_j is not degenerated. So shrinking O_j by a sufficiently small amount will not make $C_j^{(j-1)}$ and $C_j^{(j+1)}$ overlap.

2. B_j is still not in a shortest path between u and v. This is because the lengths of the paths between u and v changes continuously when O_j shrinks. So we can shrink O_j by a sufficiently small amount such that B_j is still not in a shortest path between u and v.

3. $|D_{\mathcal{O}^*}(u,v)|$ remains unchanged. This is because by Proposition 8, u, v are unobstructed in $\mathcal{O}^*$. So we can shrink O_j by a sufficiently small amount such that u, v are still unobstructed, which means that $|D_{\mathcal{O}^*}(u,v)| = ||uv||$ remains the same.

We claim that shrinking O_j by this transformation will not decrease $|P_{\mathcal{O}^*}(u,v)|$. Here is why. Shrinking O_i does not affect $\mathcal{O}^*$ except for A_j and the arcs and line segments containing b_{j-1} or b_j, including B_{j-1}, B_j, B_{j+1}, $\overline{a_{j-1}b_{j-1}}$, and $\overline{a_j b_j}$.

First consider the effect of shrinking O_j on $|A_j|$. Since B_j is not in a shortest

path, $|A_j| \leq ||a_{j-1}b_{j-1}|| + |B_j| + ||a_j b_j||$ and hence $|A_j|$ is less than half of the perimeter of O_j. Therefore, shrinking O_j by a sufficiently small amount while keeping a_{j-1} and a_j on the boundary of O_j will increase $|A_j|$.

Next consider the effect of shrinking O_j on a shortest path containing b_{j-1} or b_j. Let P be a shortest path between u and v in $\mathcal{O}^*$ containing b_j. Since P does not contain B_j, P includes A_j, $\overline{a_j b_j}$, and B_{j+1}. Let b'_j be the new location of b_j and let $\widehat{b_j b'_j}$ be arc between b_j and b'_j that becomes an extension of B_{j+1} after the transformation. So shrinking O_i increases $|B_{j+1}|$ by $|\widehat{b_j b'_j}|$ and decreases $||a_j b_j||$ by $||a_j b_j|| - ||a_j b'_j||$. Observe that

$$|\widehat{b_j b'_j}| > ||a_j b_j|| - ||a_j b'_j||. \tag{57}$$

For example, in Figure 5, b'_4 is the new location of b_4 after shrinking O_4 and $|\widehat{b_4 b'_4}| > ||a_4 b_4|| - ||a_4 b'_4||$. This implies that shrinking O_i increases $|B_{j+1}| + ||a_j b_j||$; i.e., shrinking O_i increases the length of P. By a similar argument, if P contains b_{j-1}, shrinking O_i increases the length of P.

So in any case, shrinking O_j will not decrease $|P_{\mathcal{O}^*}(u,v)|$.

In summary, by shrinking O_j we have a new chain that satisfies condition (1) and (2) of $\mathcal{O}^*$, and has a smaller sum of radii than $\mathcal{O}^*$, which is a contradiction to condition (3) of $\mathcal{O}^*$.

This proves that $B_1 B_2 \ldots B_n$ is a shortest path between u and v in $\mathcal{O}^*$. Similarly, $A_1 A_2 \ldots A_n$ is also a shortest path between u and v. This completes the proof. $\square$

Now we are ready to prove that $\Phi_{\mathcal{O}^*} \geq -\frac{\sqrt{5}\varphi}{3}|P_{\mathcal{O}^*}(u,v)|$.

See Figure 6. We align the centers $o_1 \ldots o_n$ of the circles in $\mathcal{O}^*$ on a straight line l by the following operations: first rotate O_1 around o_1 such that u comes to a location where $|A_1| = |B_1|$; then rotate O_1 and O_2 around o_2 such that $|A_2| = |B_2|$; then rotate O_1, O_2, and O_3 around o_3 such that $|A_3| = |B_3|$; and so on. Let $\overline{\mathcal{O}^*}$ be the resulting chain after the rotation operations. In $\overline{\mathcal{O}^*}$, we have $A_i = B_i$ for $1 \leq i \leq n$ and $\sum_{i=1}^{n}(|A_i| + |B_i|)$ in $\overline{\mathcal{O}^*}$ remains the same as in $\mathcal{O}^*$, which is $2|P_{\mathcal{O}^*}(u,v)|$ by Proposition 9. So the length of the path $A_1 \ldots A_n$

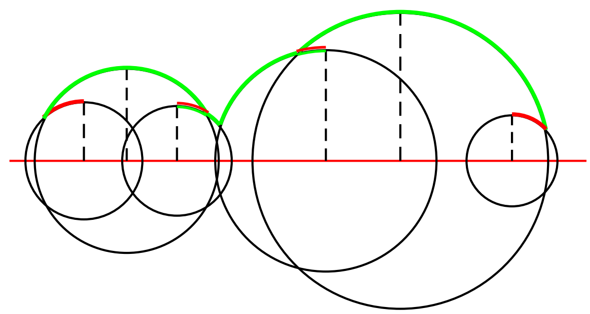

Figure 6: The centers of the circles in $\overline{\mathcal{O}^*}$ are aligned on a straight line. The path $\mathcal{P}_{\overline{\mathcal{O}^*}}$ in which the overlapping portions of red arcs and green arcs cancel each other.

in $\overline{\mathcal{O}^*}$ is

$$|A_1 \dots A_n| = |P_{\mathcal{O}^*}(u,v)|. \tag{58}$$

Recall that $\Phi_{\mathcal{O}^*} = \varphi(r_n - r_1) - \frac{\varphi}{3}\sum_{i=2}^{n}(2H_i + V_i)$. By reversing the labels of the circles in $\mathcal{O}^*$, if necessary, we can assume that $r_n \geq r_1$. Now it suffices to show that $\sum_{i=2}^{n}(2H_i + V_i) \leq \sqrt{5}|P_{\mathcal{O}^*}(u,v)|$.

Recall from Definition 2 that H_i and V_i are the horizontal and vertical distance traveled along the path $\mathcal{P}_i$, where $\mathcal{P}_i$ is a path from $q_{i-1}^{\rightarrow}$, a peak of O_{i-1}, to $q_i^{\leftarrow}$, a peak of O_i. Also recall that $\mathcal{P}_i$ consists of green or red arcs, the green arcs in $\mathcal{P}_i$ contribute positively to H_i and V_i, and the red arcs in $\mathcal{P}_i$ contribute negatively to H_i and V_i. Since the centers $o_1 \dots o_n$ are all aligned on a straight line in $\overline{\mathcal{O}^*}$, the two peaks $q_i^{\leftarrow}$ and $q_i^{\rightarrow}$ of every circle O_i overlap, for $2 \leq i \leq n-1$. So the paths $\mathcal{P}_i$ are joined at the peaks to form a single path from $q_1^{\rightarrow}$ to $q_n^{\leftarrow}$; denote it by $\mathcal{P}_{\overline{\mathcal{O}^*}}$. Refer to Figure 6. Let the overlapping portions of red arcs and green arcs in $\mathcal{P}_{\overline{\mathcal{O}^*}}$ cancel each other out and then remove the remaining red arcs, if any, at the beginning or the end of $\mathcal{P}_{\overline{\mathcal{O}^*}}$. The resulting path, denoted by $\mathcal{P}'_{\overline{\mathcal{O}^*}}$, is a subpath of $A_1 \dots A_n$. Thus from (58)

$$|\mathcal{P}'_{\overline{\mathcal{O}^*}}| \leq |A_1 \dots A_n| = |P_{\mathcal{O}^*}(u,v)|. \tag{59}$$

Let $H'_{\overline{\mathcal{O}^*}}$ and $V'_{\overline{\mathcal{O}^*}}$ be the horizontal and vertical distance traveled by $\mathcal{P}'_{\overline{\mathcal{O}^*}}$. By

the definition of $\mathcal{P}'_{\overline{\mathcal{O}^*}}$, $\sum_{i=2}^{n} H_i \le H'_{\overline{\mathcal{O}^*}}$ and $\sum_{i=2}^{n} V_i \le V'_{\overline{\mathcal{O}^*}}$. Combining these with (59), we have

$$\begin{aligned}\sqrt{(\sum_{i=2}^{n} H_i)^2 + (\sum_{i=2}^{n} V_i)^2} &\le \sqrt{(H'_{\overline{\mathcal{O}^*}})^2 + (V'_{\overline{\mathcal{O}^*}})^2} \\ &\le |\mathcal{P}'_{\overline{\mathcal{O}^*}}| \le |P_{\mathcal{O}^*}(u,v)|. \qquad (60)\end{aligned}$$

It follows that

$$\begin{aligned}2\sum_{i=2}^{n} H_i + \sum_{i=2}^{n} V_i &\le \sqrt{5(\sum_{i=2}^{n} H_i)^2 + 5(\sum_{i=2}^{n} V_i)^2} \\ &\le \sqrt{5}|P_{\mathcal{O}^*}(u,v)|, \qquad (61)\end{aligned}$$

as required. This completes the proof of Lemma 2. □

6 Conclusions

In this paper, we showed that the stretch factor of the Delaunay triangulation is less than 1.998 by proving the same upper bound on the stretch factor of a chain.

There are a few places where our approach can be further improved. First, the potential function can be improved to yield a better upper bound. For example, if we define the potential function $\Phi_{\mathcal{O}}$ to be the length of the segment of $\overline{uv}$ inside O_n, then we can improve the upper bound to 1.98, although the analysis is quite complicated. Secondly, the key components of our approach are the proofs of Proposition 6 and Proposition 7, which rely largely on functional analysis. We hope to gain insight of the underlying geometry that will help us simplify the proofs and push the upper bound closer to the tight bound, which we believe is around 1.6.

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References

[1] R. Benson. *Euclidean Geometry and Convexity.* McGraw-Hill, New York, 1966.

[2] P. Bose, L. Devroye, M. Löffler, J. Snoeyink, and V. Verma. Almost all delaunay triangulations have stretch factor greater than $\pi/2$. *Computational Geometry*, 44(2):121 – 127, 2011.

[3] P. Bose, J. Gudmundsson, and M. Smid. Constructing plane spanners of bounded degree and low weight. *Algorithmica*, 42(3-4):249–264, 2005.

[4] P. Bose and M. Smid. On plane geometric spanners: A survey and open problems, 2010. Manuscript at: people.scs.carleton.ca/∼michiel/surveyplanespanners.pdf.

[5] P. Chew. There are planar graphs almost as good as the complete graph. *Journal of Computer and System Sciences*, 39(2):205–219, 1989.

[6] S. Cui, I.A. Kanj, and G. Xia. On the stretch factor of delaunay triangulations of points in convex position. *Computational Geometry*, 44(2):104 – 109, 2011.

[7] D. Dobkin, S. Friedman, and K. Supowit. Delaunay graphs are almost as good as complete graphs. In *Proceedings of the 28th Annual Symposium on Foundations of Computer Science, FOCS*, pages 20–26, 1987.

[8] D. Dobkin, S. Friedman, and K. Supowit. Delaunay graphs are almost as good as complete graphs. *Discrete and Comp. Geom.*, 5(4):399–407, 1990.

[9] J. Gudmundsson, C. Levcopoulos, and G. Narasimhan. Fast greedy algorithms for constructing sparse geometric spanners. *SIAM Journal on Computing*, 31(5):1479–1500, 2002.

[10] I.A. Kanj, L. Perković, and G. Xia. On spanners and lightweight spanners of geometric graphs. *SIAM Journal on Computing*, 39(6):2132–2161, 2010.

[11] W. Kaplan. *Maxima and minima with applications: practical optimization and duality*. John Wiley, New York, 1999.

[12] J. Keil and C. Gutwin. The Delaunay triangulation closely approximates the complete Euclidean graph. In *Proceedings of the 1st Workshop on Algorithms and Data Structures, WADS*, pages 47–56, 1989.

[13] J. Keil and C. Gutwin. Classes of graphs which approximate the complete Euclidean graph. *Discrete and Comp. Geom.*, 7:13–28, 1992.

[14] X.-Y. Li, G. Calinescu, P.-J. Wan, and Y. Wang. Localized delaunay triangulation with application in Ad Hoc wireless networks. *IEEE Trans. Parallel and Distributed Systems*, 14(10):1035–1047, 2003.

[15] G. Narasimhan and M. Smid. *Geometric Spanner Networks*. Cambridge University Press, 2007.

[16] R.J. Vanderbei. Extension of Piyavskii's algorithm to continuous global optimization. *Journal of Global Optimization*, 14:205–216, 1997.

[17] G. Xia and L. Zhang. Improved lower bound for the stretch factor of delaunay triangulations, 2010. Manuscript. Poster in *FWCG'10: 20th Annual Fall Workshop on Computational Geometry*.

7 Appendix

7.1 Verify Inequalities (53), (54), (55), and (56)

Let $z = \alpha/4 + \sin\alpha/4$. Since $0 < z < \pi/4$, we have $1 < \frac{z}{\sin z} < 1.111$. Furthermore,

$$\frac{dz}{d\alpha} = \frac{d(\alpha/4 + \sin\alpha/4)}{d\alpha} = 1/4 + \cos\alpha/4,$$

and

$$\frac{d(\frac{z}{\sin z})}{dz} = \frac{1}{\sin z} - \frac{z\cos z}{(\sin z)^2} \leq \frac{1}{\sin z} - \frac{\cos z}{\sin z} = \tan(z/2) < 0.415.$$

Recall that

$$\gamma^+ = \frac{3\sin\alpha - \alpha}{4} + \arcsin\left(\frac{\alpha + \sin\alpha}{4\lambda\sin(\frac{\alpha+\sin\alpha}{4})}\right) = \frac{3\sin\alpha - \alpha}{4} + \arcsin(\frac{z}{\lambda\sin z}).$$

So

$$\begin{aligned}
\left|\frac{d(\gamma^+)}{d\alpha}\right| &\leq \left|\frac{3\cos\alpha - 1}{4} + \frac{1}{\lambda\sqrt{1 - (\frac{z}{\lambda\sin z})^2}}\frac{d(\frac{z}{\sin z})}{dz}\frac{dz}{d\alpha}\right| \\
&\leq 1 + \left|\frac{0.415}{\lambda\sqrt{1 - (\frac{z}{\lambda\sin z})^2}}(1/4 + \cos\alpha/4)\right| \\
&\leq 1 + \left|\frac{0.415}{\lambda\sqrt{1 - (\frac{1.111}{\lambda})^2}}(1/4 + 1/4)\right| \\
&< 1.15.
\end{aligned}$$

Recall that

$$f(\alpha, \beta, \gamma) = -\lambda(\cos\gamma - \cos\alpha(\cos(\beta - \gamma) + \beta\sin(\beta - \gamma))). \tag{62}$$

We have

$$\left|\frac{df(\alpha, 0, 0)}{d\alpha}\right| = \left|\frac{d(-\lambda(1 - \cos\alpha))}{d\alpha}\right| = |-\lambda\sin\alpha| < 2.$$

$$
\begin{aligned}
\left|\frac{df(\alpha,0,\gamma^+)}{d\alpha}\right| &= \left|\frac{d(-\lambda(\cos\gamma^+ - \cos\alpha\cos(-\gamma^+)))}{d\alpha}\right| \\
&= \left|\frac{d(-\lambda\cos\gamma^+(1-\cos\alpha))}{d\alpha}\right| \\
&\leq \lambda(|\frac{d(\gamma^+)}{d\alpha}\sin\gamma^+(1-\cos\alpha)| + |\sin\alpha\cos\gamma^+|) \\
&\leq 1.8(1.15+1) < 4.
\end{aligned}
$$

$$
\begin{aligned}
\left|\frac{df(\alpha,\sin\alpha,0)}{d\alpha}\right| &= \left|\frac{d(-\lambda(1-\cos\alpha\cos(\sin\alpha) - \cos\alpha\sin\alpha\sin(\sin\alpha)))}{d\alpha}\right| \\
&\leq \lambda(|\sin\alpha\cos(\sin\alpha)| + |(\cos\alpha)^2\sin\alpha\cos(\sin\alpha)| + |(\sin\alpha)^2\sin(\sin\alpha)|) \\
&\leq 1.8(1+1+1) < 6.
\end{aligned}
$$

$$
\begin{aligned}
\left|\frac{df(\alpha,\sin\alpha,\gamma^+)}{d\alpha}\right| &= \left|\frac{d(-\lambda(\cos\gamma^+ - \cos\alpha\cos(\sin\alpha-\gamma^+) - \cos\alpha\sin\alpha\sin(\sin\alpha-\gamma^+)))}{d\alpha}\right| \\
&= \left|\frac{d(-\lambda(\cos\gamma^+ - \cos\alpha\cos(\sin\alpha-\gamma^+) - \frac{\sin(2\alpha)}{2}\sin(\sin\alpha-\gamma^+)))}{d\alpha}\right| \\
&\leq \lambda(|\frac{d(\gamma^+)}{d\alpha}\sin\gamma^+| + |\sin\alpha\cos(\sin\alpha-\gamma^+)| \\
&\quad + |\cos\alpha\sin(\sin\alpha-\gamma^+)(\cos\alpha - \frac{d(\gamma^+)}{d\alpha})| \\
&\quad + |\cos(2\alpha)\sin(\sin\alpha-\gamma^+)| + |\frac{\sin(2\alpha)}{2}\cos(\sin\alpha-\gamma^+)(\cos\alpha - \frac{d(\gamma^+)}{d\alpha})|) \\
&< 1.8(1.15+1+(1+1.15)+1+(1+1.15)/2) < 12.
\end{aligned}
$$

Also

$$
\begin{aligned}
\left|\frac{d(\sin\alpha - \alpha\cos\alpha - \frac{2\varphi}{3} - \frac{2\varphi}{3}\cos\alpha)}{d\alpha}\right| &= |\alpha\sin\alpha + \frac{2\varphi}{3}\sin\alpha| \leq \pi + \frac{2\varphi}{3} < 4. \\
\left|\frac{d(\sin\alpha - \alpha\cos\alpha - \frac{2\varphi}{3} - \frac{4\varphi}{3}\cos\alpha)}{d\alpha}\right| &= |\alpha\sin\alpha + \frac{4\varphi}{3}\sin\alpha| \leq \pi + \frac{4\varphi}{3} < 4.
\end{aligned}
$$

Therefore the Lipschitz constants are

$$\left|\frac{dg_1(\alpha)}{d\alpha}\right| = \left|\frac{d(\sin\alpha - \alpha\cos\alpha - \frac{2\varphi}{3} - \frac{2\varphi}{3}\cos\alpha)}{d\alpha}\right| + \left|\frac{df(\alpha,0,0)}{d\alpha}\right| < 4+2 < 16.$$

$$\left|\frac{dg_2(\alpha)}{d\alpha}\right| = \left|\frac{d(\sin\alpha - \alpha\cos\alpha - \frac{2\varphi}{3} - \frac{2\varphi}{3}\cos\alpha)}{d\alpha}\right| + \left|\frac{df(\alpha,0,\gamma^+)}{d\alpha}\right| < 4+4 < 16.$$

$$\left|\frac{dg_3(\alpha)}{d\alpha}\right| = \left|\frac{d(\sin\alpha - \alpha\cos\alpha - \frac{2\varphi}{3} - \frac{2\varphi}{3}\cos\alpha)}{d\alpha}\right| + \left|\frac{df(\alpha,\sin\alpha,0)}{d\alpha}\right| < 4+6 < 16.$$

$$\left|\frac{dg_4(\alpha)}{d\alpha}\right| = \left|\frac{d(\sin\alpha - \alpha\cos\alpha - \frac{2\varphi}{3} - \frac{2\varphi}{3}\cos\alpha)}{d\alpha}\right| + \left|\frac{df(\alpha,\sin\alpha,\gamma^+)}{d\alpha}\right| < 4+12 = 16$$

Now we can use a simplified version of the Piyavskii's algorithm [16] for Lipschitz optimization to verify that $g_i(\alpha) < 0$, $1 \le i \le 4$. The following Algorithm **Bound**(g_i, s, t) will either find a value $g_i(\alpha) \ge 0$ in the given range or return an upper bound on the value of g_i in range $[s, t]$ that is less than 0. We run **Bound**(g_i, s, t) with s and t set to be the lower and upper bound on the range of α and verify that the upper bound of $g_i(\alpha)$ is indeed less than 0 in that range, for $1 \le i \le 4$, where $g_i(0)$ is set to be $\lim_{\alpha\to 0} g_i(\alpha)$.

Algorithm **Bound**(g_i, s, t)**, for** $1 \le i \le 4$

1. **if** $g_i(s) \ge 0$ or $g_i(t) \ge 0$ **return** $\max\{g_i(s), g_i(t)\}$.
2. apex $= \max\{g_i(s), g_i(t)\} + 16 * (t-s)/2$
3. **if** apex ≥ 0, **do**:
   3.1. apex $= \max\{$**Bound**$(g_i, s, (s+t)/2)$, **Bound**$(g_i, (s+t)/2, t)\}$
4. **return** apex